TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 323, Number 2, February 1991

A NONSTANDARD RESONANCE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

SHAIR AHMAD

ABSTRACT. Necessary and sufficient conditions are established for the existence of bounded solutions for a class of second order differential equations.

1. INTRODUCTION AND SUMMARY

This paper is a study of a Landesman-Lazer type problem for an ordinary differential equation in which the usual methods of nonlinear analysis such as fixed point theory, degree theory, monotone operator theory, and critical point theory do not seem to be applicable.

We consider the differential equation

$$y''(t) + cy'(t) + g(y(t)) = p(t) \tag{1.1}$$

where p and g are continuous functions defined on $(-\infty, \infty)$ satisfying certain conditions given below, and c is a constant. We shall study the problem of the existence of solutions of (1.1) that are bounded on a ray $t_0 \le t < \infty$ and those that are bounded on $(-\infty, \infty)$.

In order to motivate our main results, we first recall some known results for the case where p is periodic. Suppose that the period of p is $L, L > 0$, and let

$$p_0 = \frac{1}{L}\int_0^L p(t)\,dt.$$

It follows from a result of Lazer [6] that if

$$g(\xi)/\xi \to 0 \quad \text{as } |\xi| \to \infty, \tag{1.2}$$

and there exists $\xi_1, \xi_1 \ge 0$, such that

$$(g(\xi) - p_0)\xi \ge 0 \quad \text{if } |\xi| \ge \xi_1 \tag{1.3}$$

then for any value of c—even for $c = 0$—there exists at least one L-periodic solution of (1.1). Independently, Bebernes and Martelli [1] and Ward [11] established results which show that if $c \neq 0$, then condition (1.3) alone guarantees the existence of a T-periodic solution of (1.1). This was also contained in the

Received by the editors February 14, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34A34; Secondary 34C11.

work of N. Rouche and J. Mawhin [8], where they studied a more general system of differential equations. The proofs of all of the above-cited results use some method of compactness.

In this paper we shall consider nonlinearities g of the type where the limits

$$\lim_{\xi\to\infty} g(\xi) = g(\infty), \qquad \lim_{\xi\to-\infty} g(\xi) = g(-\infty)$$

exist in the broad sense that they are possibly infinite and that for $\xi \in (-\infty, \infty)$

$$g(-\infty) \le g(\xi) \le g(\infty). \tag{1.4}$$

In engineering literature such functions model saturation (see §6).

If $p(t)$ is L-periodic and p_0 is defined as above, it follows that the condition

$$g(-\infty) < p_0 < g(\infty), \tag{1.5}$$

usually referred to as a Landesman-Lazer condition, implies (1.3), and is therefore sufficient for the existence of an L-periodic solution of (1.1) if $c \neq 0$. Conversely, if one assumes that (1.1) has an L-periodic solution $y_0(t)$, it follows from integrating the differential equation from 0 to L that

$$\frac{1}{L}\int_0^L g(y_0(t))dt = p_0. \tag{1.6}$$

Therefore, assuming (1.4), this implies that the condition

$$g(-\infty) \le p_0 \le g(\infty) \tag{1.7}$$

is necessary for the existence of an L-periodic solution of (1.1). If (1.4) is replaced by the stronger condition that for all ξ, $\xi \in (-\infty, \infty)$,

$$g(-\infty) < g(\xi) < g(\infty) \tag{1.4$'$}$$

then (1.6) implies that (1.5) holds. Therefore, if $(1.4)'$ holds and $c \neq 0$, then (1.5) is both necessary and sufficient for the existence of an L-periodic solution.

In this paper we always assume that either (1.4) or $(1.4)'$ holds and that $c > 0$. Results for which $c < 0$ and for which the inequalities in (1.4) and $(1.4)'$ are reversed can also be obtained, but these conditions do not seem to have physical meaning. We replace the condition that $p(t)$ be periodic by the assumption that $p(t)$ be continuous and bounded on $(-\infty, \infty)$ and that there exist a number p_0 such that for any $a \in (-\infty, \infty)$

$$\lim_{T\to\infty} \int_a^{a+T} p(t)\,dt = p_0 \tag{1.8}$$

uniformly with respect to a. If $p(t)$ happens to be L-periodic, then this condition holds and p_0 is the same number as defined earlier. More generally, condition (1.8) will hold if $p(t)$ is almost periodic (see [2]), but this is not necessary. For example, (1.8) will hold if $p(t)$ is the sum of an almost periodic function and a function in $L^1(-\infty, \infty)$. Our main result shows that given $t_0 \in (-\infty, \infty)$, if (1.4) holds then the strict inequalities (1.5) are sufficient and the weak inequalities are necessary for the existence of a solution *bounded* on $[t_0, \infty)$. Moreover, if $(1.4)'$ holds then the inequalities (1.5) are both necessary and sufficient for the existence of a solution bounded on $[t_0, \infty)$.

Unlike any of the above-cited references, we show that conditions (1.5) have an implication for the totality of solutions of (1.1) defined on $[t_0, \infty)$: namely, they are *all* bounded on this interval. Also, we show that (1.5) implies the existence of a solution of (1.1) which is bounded on $(-\infty, \infty)$.

Example. From the above-quoted theorems one might conjecture that if $p(t)$ satisfies the above conditions and there exists $\xi_1 \geq 0$ such that (1.3) holds, then there exists a solution of (1.1) bounded on $(-\infty, \infty)$. However, this is false for the following reason: There exists an almost periodic function $p(t)$ such that $p_0 = 0$, but the integral $\int_0^t p(s)\,ds$ is unbounded on $(-\infty, \infty)$ (see [3, p. 72]). If $g(\xi) \equiv 0$, then (1.4) holds and hence, by what is shown in the next section, for any solution $y(t)$ of $y''(t) + cy'(t) = p(t)$, $y'(t)$ is bounded on $[0, \infty)$ provided $c > 0$. Since

$$y'(t) + cy(t) = y'(0) + cy(0) + \int_0^t p(s)\,ds$$

it follows that $y(t)$ cannot be bounded on $[0, \infty)$.

As a byproduct of some of the estimates needed to prove our main theorem we show that if g is an arbitrary continuous function and there exists an isolated zero ξ_0 of the equation $g(\xi) - p_0 = 0$ such that $(g(\xi) - p_0)(\xi - \xi_0) > 0$ for $\xi \neq \xi_0$ and $|\xi - \xi_0|$ small, then for any $\alpha > 0$ there exist numbers $\gamma_1 > 0$, $\gamma_2 > 0$ and $c^* > 0$ such that if $t_0 \in (-\infty, \infty)$, $c \geq c^*$, and $y(t)$ is a solution of (1.1) satisfying the inequalities $|y(t_0) - p_0| < \gamma_1$ and $|y'(t_0)| < \gamma_2$, then $|y(t) - p_0| < \alpha$ and $|y'(t)| < \alpha$ for $t \geq t_0$.

It follows from a theorem due to Reuter [9] that if $g(-\infty) = -\infty$, $g(\infty) = \infty$ and p is only assumed to be continuous and bounded on $[t_0, \infty)$, then all solutions of (1.1) are bounded on $[t_0, \infty)$. However, we include the case $g(\pm\infty) = \pm\infty$ in our discussion for the sake of completeness.

Another nonstandard resonance problem was considered by Frederickson and Lazer in [4], in which it was assumed that the restoring force be linear and the damping nonlinear. They showed that if $f \geq 0$, $f \not\equiv 0$, f is continuous on $(-\infty, \infty)$ and $p(t)$ is almost periodic, then a necessary and sufficient condition that

$$y''(t) + f(y(t))y'(t) + y(t) = p(t)$$

have an almost periodic solution is that

$$F(\infty) - F(-\infty) > 2\sqrt{a_1^2 + b_1^2}$$

where $F(0) = 0$, $F'(\xi) = f(\xi)$, and

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} = \lim_{T\to\infty} \frac{1}{T} \int_0^T p(t) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix} dt.$$

Although our methods and those of [4] both involve a basic theory of analysis and differential equation, the proofs as well as the equations considered are fundamentally different.

We emphasize that since g is only assumed to be continuous, the solutions of the initial value problem associated with (1.1) are, in general, not unique. However, this does not introduce any technical complications.

2. Boundedness of the derivative

In this section we assume that p and g are defined and continuous on $(-\infty, \infty)$, that g has limits $g(\pm\infty)$, which are possibly infinite, that p is bounded, and

$$g(-\infty) \leq g(\xi) \leq g(\infty), \qquad -\infty < \xi < \infty. \tag{2.1}$$

Let $c_1 > 0$ be a constant such that

$$|p(t)| \leq c_1, \qquad t \in (-\infty, \infty). \tag{2.2}$$

We consider the differential equation

$$y''(t) + cy'(t) + g(y(t)) = p(t) \tag{2.3}$$

where it is always assumed that $c > 0$. If $t_0 \in (-\infty, \infty)$ and y_0 and v_0 are given numbers, then by a solution of the initial value problem for (2.3) determined by the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = v_0 \tag{2.4}$$

we mean a solution whose interval of existence is maximal.

Theorem 2.1. *For arbitrary t_0, y_0, and v_0, the interval of existence of the initial value problem given by* (2.3) *and* (2.4) *contains* $[t_0, \infty)$. *There exists $k > 0$ such that for any $A > 0$ and $B > 0$ there exists $T(A, B) > 0$ such that if $t_0 \in (-\infty, \infty)$ and $y(t)$ is a solution of the initial value problem* (2.3)–(2.4) *and*

$$|y_0| \leq A, \qquad |v_0| \leq B \tag{2.5}$$

then

$$t \geq t_0 + T(A, B) \Rightarrow |y'(t)| \leq k. \tag{2.6}$$

Given $A > 0$ and $B > 0$, there exists $b(A, B) > 0$ such that if $t_0 \in (-\infty, \infty)$ and $y(t)$ is a solution of problem (2.3)–(2.4) *and conditions* (2.5) *hold, then*

$$|y'(t)| \leq b(A, B), \qquad t \geq t_0. \tag{2.7}$$

Proof. We defer the part of the proof concerning the interval of existence of solutions of (2.3)–(2.4) to the end. Until then we assume that y is a solution of (2.3) which exists on $[t_0, \infty)$.

There are four cases to consider.

Case I. Suppose that

(2.8)[1] $$-\infty < g(-\infty) \leq g(\infty) < \infty.$$

If c_2, $c_2 > 0$, is a constant such that $g(\xi) \le c_2$ for all $\xi \in (-\infty, \infty)$, and $c_3 = c_1 + c_2$, then it follows from (2.2) and (2.3) that $-c_3 \le y''(t) + cy'(t) \le c_3$ for all $t \ge t_0$. Multiplying these inequalities by e^{ct}, integrating from t_0 to t, and making some obvious manipulations, we obtain the inequalities

$$(-c_3/c)(1 - e^{-c(t-t_0)}) \le y'(t) - y'(t_0)e^{-c(t-t_0)} \le (c_3/c)(1 - e^{-c(t-t_0)})$$

from which it follows that if $|y'(t_0)| \le B$, then

$$|y'(t)| \le \max\{B, c_3/c\} \equiv b(B), \qquad t \ge t_0, \tag{2.9}$$

which proves (2.7). The same inequalities imply that for any $B > 0$, there exists a number $T = T(B)$ such that if $|y'(t_0)| \le B$, then

$$t \ge t_0 + T(B) \Rightarrow |y'(t)| \le 2c_3/c \tag{2.10}$$

which is (2.6) with $k = 2c_3/c$.

Case II. Suppose that

$$-\infty < g(-\infty) < g(\infty) = \infty. \tag{2.8^2}$$

Then there exists a number $c_3 > 0$ such that $p(t) - g(\xi) \le c_3$ for all $t \in (-\infty, \infty)$ and all $\xi \in (-\infty, \infty)$. Consequently, $y''(t) + cy'(t) \le c_3$ for all $t \ge t_0$. An argument similar to that in Case I shows that

$$y'(t) \le y'(t_0)e^{-c(t-t_0)} + (c_3/c)(1 - e^{-c(t-t_0)}), \qquad t \ge t_0.$$

As in Case I, this implies that if $y'(t_0) \le B$, then there exists a number $T_1 = T_1(B)$ such that

$$t \ge t_0 + T_1(B) \Rightarrow y'(t) \le 2c_3/c \tag{2.11}$$

and

$$y'(t) \le \max\{B, c_3/c\} \equiv d_1(B), \qquad t \ge t_0. \tag{2.12}$$

To prove the assertions of the theorem in Case II, we make use of some lemmas.

Let ξ_0 be chosen so large that

$$\xi \ge \xi_0 \Rightarrow g(\xi) \ge c_1 + 1. \tag{2.13}$$

Lemma 2.1. *If* $y(t_0) \le A$ *and* $y'(t_0) \le B$, *then*

$$y(t) \le \max\{A, \xi_0\} + \tfrac{1}{c}d_1(B) \equiv d_2(A, B), \qquad t \ge t_0. \tag{2.14}$$

Proof. Assuming the contrary, there exist numbers t_1 and t_2 with $t_0 \le t_1 < t_2$ such that $y(t_1) = \max\{A, \xi_0\}$, $y(t_2) = y(t_1) + \frac{1}{c}d_1(B)$, and $y(t_1) < y(t) < y(t_2)$ for $t_1 \le t \le t_2$. Since $y''(t) + cy'(t) = p(t) - g(y(t)) \le -1$ for $t_1 \le t \le t_2$, we have

$$y'(t_2) - y'(t_1) + c[y(t_2) - y(t_1)] < 0. \tag{2.15}$$

Clearly, $y'(t_2) \geq 0$, and hence from (2.12) and (2.15) we obtain $y(t_2) < y(t_1) + \frac{1}{c}y'(t_1) \leq y(t_1) + \frac{1}{c}d_1(B)$, which is a contradiction. This proves the lemma.

According to $(2.8)^2$, $c_4 = c_4(A, B) > 0$ can be chosen so that if $-\infty < \xi \leq d_2(A, B)$, then $|g(\xi)| \leq c_4(A, B)$. Therefore, if $c_5 = c_5(A, B) = c_4(A, B) + c_1$, then it follows from Lemma 2.1 that $y(t_0) \leq A$ and $y'(t_0) \leq B$ imply that $-c_5 \leq y''(t) + cy'(t) \leq c_5$ for all $t \geq t_0$. As in Case I, it follows that if $y(t_0) \leq A$ and $|y'(t_0)| \leq B$, then

(2.16) $$|y'(t)| \leq \max\{B, c_5(A, B)/c\} = b(A, B), \qquad t \geq t_0.$$

This establishes (2.7) in Case II.

In order to establish (2.6), we use

Lemma 2.2. *If* $y(t_0) \leq A$, $y'(t_0) \leq B$, *and*

$$L = L(A, B) = 1 + (c+1)(d_2(A, B) - \xi_0) + d_1(B),$$

then every subinterval of $[t_0, \infty)$ *of length* L *contains a point* t^* *such that* $y(t^*) \leq \xi_0$.

Proof. Suppose that $t_1 \in [t_0, \infty)$ and, contrary to the assumption of the lemma, $y(t) > \xi_0$ for all t in $[t_1, t_1 + L]$. For t in this range we have $y''(t) + cy'(t) = p(t) - g(y(t)) \leq -1$, and hence, by integration,

$$y'(t) \leq y'(t_1) + c(y(t_1) - y(t)) - (t - t_1).$$

From (2.12) and (2.14) we have

$$y'(t) \leq d_1(B) + c(d_2(A, B) - \xi_0) - (t - t_1)$$

for $t \in [t_1, t_1 + L]$. Setting $\bar{t} = t_1 + d_1(B) + c(d_2(A, B) - \xi_0) + 1$, it follows that for $\bar{t} \leq t \leq t_1 + L$ we have $y'(t) \leq -1$. Hence,

$$y(t_1 + L) \leq y(\bar{t}) - (t_1 + L - \bar{t}) \leq d_2(A, B) - (t_1 + L - \bar{t}) = \xi_0,$$

which is a contradiction. This completes the proof of the lemma.

Let $A > 0$, $B > 0$, and assume that $y(t_0) \leq A$, and $y'(t_0) \leq B$. From (2.11) we have $y'(t) \leq 2c_3/c$ for $t \geq t_0 + T_1(B)$, and, by what we just established above, there exists a number t^*, $t^* \in [t_0 + T_1(B), t_0 + T_1(B) + L(A, B)]$ such that $y(t^*) \leq \xi_0$. Applying Lemma 2.1, with t_0 replaced by t^*, it follows that

$$y(t) \leq \xi_0 + \frac{1}{c}d_1\left(\frac{2c_3}{c}\right) = \xi_0 + \frac{2c_3}{c^2}$$

for all $t \geq t^*$. Consequently, if

$$c_6 = \sup\{g(\xi)| - \infty < \xi < \xi_0 + 2c_3/c^2\} < \infty$$

and $c_7 = c_1 + c_6$, then $-c_7 \leq y''(t) + cy(t) < c_7$ for $t \geq t^*$. Using an argument similar to that in Case I, we obtain the inequalities

$$(-c_7/c)(1 - e^{-c(t-t^*)}) \leq y'(t) - y'(t_0)e^{-c(t-t^*)} \leq (c_7/c)(1 - e^{-c(t-t^*)}).$$

It is easy to see that for $t \geq t^*$ there exists a number $T_2 = T_2(A, B)$ such that $|y'(t)| \leq 2c_7/c$ for all $t \geq t^* + T_2(A, B)$. Recalling that $t_0 < t^* < t_0 + T_1(B) + L(A, B)$, it follows that if we set $T(A, B) = T_1(B) + T_2(A, B) + L(A, B)$, then

$$t \geq t_0 + T(A, B) \Rightarrow |y'(t)| \leq 2c_7/c. \tag{2.17}$$

This proves (2.6) with $k = 2c_7/c$.

Case III. Suppose that

$$-\infty = g(-\infty) < g(\xi) \leq g(\infty) < \infty, \qquad \xi \in (-\infty, \infty). \tag*{(2.8)^3}$$

If we set $\hat{g}(\xi) = -g(-\xi)$, then for $\xi \in (-\infty, \infty)$ we have $-\infty < \hat{g}(-\infty) \leq g(\xi) < \hat{g}(\infty) = \infty$. By what we showed in Case II, there exists a number k such that if $t_0 \in (-\infty, \infty)$, $A > 0$, $B > 0$, and $\hat{y}(t)$ is defined for $t_0 \leq t < \infty$ and satisfies

$$\hat{y}''(t) + c\hat{y}'(t) + \hat{g}(\hat{y}(t)) = -p(t) \tag{2.18}$$

and if $|\hat{y}(t_0)| \leq A$, $|\hat{y}'(t_0)| \leq B$, then $t \geq t_0 \Rightarrow |\hat{y}'(t_0)| \leq b(A, B)$, and $t \geq t_0 + T(A, B) \Rightarrow |\hat{y}'(t)| \leq k$.

If y is defined on $[t_0, \infty)$ and satisfies $y'' + cy' + g(y) = p(t)$, then $\hat{y}(t) \equiv -y(t)$ will satisfy (2.18). Hence, if $|y(t_0)| \leq A$ and $|y'(t_0)| \leq B$, then (2.6) holds for $t \geq t_0 + T(A, B)$, and (2.7) holds for $t \geq t_0$.

Case IV. Suppose that

$$-\infty = g(-\infty) < g(\xi) < g(\infty) = \infty, \qquad \xi \in (-\infty, \infty). \tag*{(2.8)^4}$$

In this case we can prove Theorem 2.1 by using a Liapunov function (see, e.g. [10 or 12]).

If $G(\xi)$ is defined by $G'(\xi) = g(\xi)$, and $G(0) = 0$, then, by L'Hospital's rule,

$$G(\xi)/|\xi| \to \infty \quad \text{as } |\xi| \to \infty. \tag{2.19}$$

Let $\vec{F}(y, v, t)$ be the time dependent vector field given by

$$\vec{F}(y, v, t) = \operatorname{col}(v, p(t) - cv - g(y)),$$

which corresponds to the first order system equivalent to the second order differential equation (2.3).

If for $v \geq 2c_1/c$ and $-\infty < y < \infty$, we set

$$E_1(y, v) = v^2/2 + G(y) + 2c_1 y/3, \tag{2.20}$$

then for $(y, v, t) \in (-\infty, \infty) \times [2c_1/c, \infty) \times (-\infty, \infty)$

$$\begin{aligned} \operatorname{grad} E_1(y, v) \cdot \vec{F}(y, v, t) &= -v[cv - p(t) - 2c_1/3] \\ &\leq -(2c_1/c)[2c_1 - c_1 - 2c_1/3] = -2c_1^2/3c < 0. \end{aligned} \tag{2.21}$$

Similarly, if for $v \leq -2c_1/c$ and $-\infty < y < \infty$, we set

$$E_2(y, v) = v^2/2 + G(y) - 2c_1 y/3, \tag{2.22}$$

then for $(y, v, t) \in (-\infty, \infty) \times (-\infty, -2c_1/c] \times (-\infty, \infty)$

$$\begin{aligned}\operatorname{grad} E_2(y, v) \cdot \vec{F}(y, v, t) &= v[-cv + p(t) - 2c_1/3] \\ &\leq -(2c_1/c)[2c_1 - c_1 - 2c_1/3] = -2c_1^2/3c.\end{aligned} \tag{2.23}$$

If $-2c_1/c \leq v \leq 2c_1/c$ and $-\infty < y < \infty$, we let

$$E_3(y, v) = v^2/2 + G(y) + \tfrac{c}{3}yv. \tag{2.24}$$

Since $(2.8)^4$ implies that $(\operatorname{sgn} y)\, g(y) \to \infty$ as $|y| \to \infty$, we may choose $\bar{a} > 0$ so large that if $|y| \geq \bar{a}$, then $-cyg(y)/3 + c_1c|y| + 2c_1^2/c \leq -1$. It follows that if $|v| \leq 2c_1/c$, $|y| \geq a$, and $t \in (-\infty, \infty)$, then

$$\begin{aligned}\operatorname{grad} E_3(y, v) \cdot \vec{F}(y, v, t) &= vp(t) - 2cv^2/3 + cy(p(t) - cv - g(y))/3 \\ &\leq -cyg(y)/3 + c_1c|y| + 2c_1^2/c \leq -1.\end{aligned} \tag{2.25}$$

If $E(y, v)$ is the continuous function defined by

$$E(y, v) = \begin{cases} E_1(y, v) & \text{if } v \geq 2c_1/c,\ y \in R, \\ E_2(y, v) & \text{if } v \leq -2c_1/c,\ y \in R, \\ E_3(y, v) & \text{if } |v| < 2c_1/c,\ y \in R. \end{cases}$$

Then, from (2.19) we see that

$$E(y, v) \to \infty \quad \text{as } y^2 + v^2 \to \infty. \tag{2.26}$$

We define $E' : R^3 \to R$ by setting

$$E'(y_0, v_0, t_0) = \limsup_{s \to 0^+} \frac{1}{s}[E(y(t_0 + s), y'(t_0 + s) - E(y_0, v_0))]$$

where the lim sup as $s \to 0^+$ is taken over all solutions of (2.3) such that $y(t_0) = y_0$ and $y'(t_0) = v_0$. According to (2.21), (2.23), and (2.25) if K_0 denotes the set of points (y, v) in R^2 such that $|y| \leq \bar{a}$ and $|v| \leq 2c_1/c$, then

$$(y, v) \notin K_0 \Rightarrow E'(y, v, t) \leq -\delta < 0 \tag{2.27}$$

for arbitrary $t \in R$, where $\delta = \min\{1, -2c_1^2/3c\}$. Let $A > 0$, $B > 0$, and let $y(t)$ be a solution of (2.3) with $|y(t_0)| \leq A$ and $|y'(t_0)| \leq B$. Let K_1 be the rectangle consisting of points (y, v) such that $|y| \leq A_1 \equiv \max\{\bar{a}, A\}$ and $|v| \leq B_1 \equiv \max\{2c_1/c, B\}$. By (2.19), there exist numbers $a(A, B)$ and $b(A, B)$ such that if K_2 denotes the rectangle consisting of points (y, v) such that $|y| \leq a(A, B)$ and $|v| \leq b(A, B)$, then the minimum of E on the boundary of K_2 is greater than the maximum of E on K_1. By standard theory of Liapunov functions (see [9, pp. 371–378; 12, Chapter II]) it follows from (2.27) that if $y(t)$ is a solution of (2.3) such that $(y(t_0), y'(t_0)) \in K_1$, then $(y(t), y'(t)) \in$ Interior K_2 for $t \geq t_0$. Therefore, if $|y(t_0)| \leq A$ and $|y'(t_0)| \leq B$, then $|y(t)| \leq a(A, B)$, $|y'(t)| \leq b(A, B)$ for $t \geq t_0$, and this gives (2.7).

The same reasoning shows that if k_1 and k are so large that the minimum of $E(y, v)$ on the boundary of the rectangle $K_0' = \{(y, v) \mid |y| \leq k_1, |v| \leq k\}$

is bigger than the maximum of $E(y, v)$ on K_0, then if $y(t)$ is a solution of (2.3) with $(y(t_0), y'(t_0)) \in K_0$, it follows that $(y(t), y'(t)) \in K_0'$ for $t \geq t_0$.

Let $A > 0$, $B > 0$, and let K_1 and K_2 be defined as above. If $y(t)$ is a solution of (2.3) with $(y(t_0), y'(t_0)) \in K_1$, then $(y(t), y'(t)) \in K_2$ for $t \geq t_0$. It follows from (2.27) that if

$$T(A, B) \equiv \left(\max_{K_2} E - \min_{K_2} E\right) / \delta,$$

then there exists a number $t_* \in [t_0, t_0 + T(A, B)]$ such that $(y(t_*), y'(t_*)) \in K_0$. Since this implies that $(y(t), y'(t)) \in K_0'$, we see that if $|y(t_0)| \leq A$ and $|y'(t_0)| \leq B$, then $|y(t)| \leq k_1$ and $|y'(t)| \leq k$ if $t \geq t_0 + T(A, B)$, which gives (2.6).

Up to this point we have assumed that a solution of the initial value problem (2.4) is defined on $[t_0, \infty)$. If in the above arguments we replace this assumption with the assumption that $y(t)$ is defined on $[t_0, \alpha)$, where $\alpha < \infty$, then the same reasoning shows that given $A > 0$ and $B > 0$, there exists $b(A, B)$ such that if $|y(t_0)| \leq A$ and $|y'(t_0)| \leq B$, then $|y'(t)| \leq b(A, B)$ for $t_0 \leq t < \alpha$. This implies that $|y(t)| \leq A + (t - t_0)b(A, B)$ for $t_0 \leq t < \alpha$. It follows from standard theory that this implies that $y(t)$ is defined on $[t_0, \infty)$, and the proof of Theorem 2.1 is complete.

3. The main theorem

In this section we prove

Theorem 3.1. *Let $p(t)$ be defined and continuous for $-\infty < t < \infty$. Assume that there exists $c_1 > 0$ such that $|p(t)| \leq c_1$ for all $t \in R$ and that there exists a number p_0 such that for any $a \in R$*

$$\lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} p(t)\, dt = p_0 \tag{3.1}$$

where the limit exists uniformly with respect to $a \in R$. Suppose that $g: R \to R$ is continuous and (2.1) holds. Let $c > 0$. If $t_0 \in (-\infty, \infty)$, then in order that there exist a solution of (2.3) which is bounded on $[t_0, \infty)$ it is necessary that

$$g(-\infty) \leq p_0 \leq g(+\infty) \tag{3.2}$$

and sufficient that

$$g(-\infty) < p_0 < g(+\infty). \tag{3.3}$$

If

$$\xi \in (-\infty, \infty) \Rightarrow g(-\infty) < g(\xi) < g(\infty) \tag{3.4}$$

then inequalities (3.3) are both necessary and sufficient for the existence of a solution which is bounded on $[t_0, \infty)$. If inequalities (3.3) hold, then given any $A > 0$ and $B > 0$, there exist numbers $a(A, B)$ and $b(A, B)$ such that if

$t_0 \in (-\infty, \infty)$ and $y(t)$ is a solution of (2.3) *with* $|y(t_0)| \leq A$ *and* $|y'(t_0)| \leq B$, *then*

$$|y(t)| \leq a(A, B) \quad \text{and} \quad |y'(t)| \leq b(A, B), \qquad t \geq t_0. \tag{3.5}$$

Moreover, there exist fixed numbers k_1 and k such that given $A > 0$ and $B > 0$, there exists $T_0(A, B)$ such that if $|y(t_0)| \leq A$ and $|y'(t_0)| \leq B$, then

$$t \geq t_0 + T_0(A, B) \Rightarrow |y(t)| \leq k_1 \quad \text{and} \quad |y'(t)| \leq k. \tag{3.6}$$

Proof. We first prove the necessity of conditions (3.2) and (3.3) under the assumptions (2.1) and (3.4), respectively.

Assume that conditions (2.1) hold and that $y(t)$ is a solution of (2.3) which is bounded on $[t_0, \infty)$. As we proved in the previous section, $y'(t)$ is also bounded on $[t_0, \infty)$. Therefore,

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} (y''(t) + cy'(t))\, dt \\ &\quad = \lim_{T\to\infty} \frac{1}{T}[y'(t_0 + T) - y'(t_0) + c(y(t_0 + T) - y(t_0))] = 0. \end{aligned}$$

It follows from (2.3) that

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} [p(t) - g(y(t))]\, dt = 0$$

and therefore,

$$P_0 = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} g(y(t))\, dt. \tag{3.7}$$

Since conditions (2.1) imply that for all $T > 0$

$$g(-\infty) \leq \frac{1}{T} \int_{t_0}^{t_0+T} g(y(t))\, dt \leq g(\infty)$$

we obtain (3.2).

Suppose now that the stronger condition (3.3) holds. Again, let $y(t)$ be a solution bounded on $[t_0, \infty)$, and let $r > 0$ be chosen so that $|y(t)| \leq r$ for all $t \geq t_0$. If d_* and d^* denote the minimum and maximum of $g(\xi)$ for $-r \leq \xi \leq r$ respectively, then from (2.4) we have $g(-\infty) < d_* \leq g(y(t)) \leq d^* < g(\infty)$ for all $t \geq t_0$. The same argument given above shows that (3.7) holds. Therefore, since for all $T > 0$ we have

$$d_* \leq \frac{1}{T} \int_{t_0}^{t_0+T} g(y(t))\, dt \leq d^*$$

it follows that $d_* \leq p_0 \leq d^*$, and this gives inequalities (3.3); and thus the necessary part of the theorem is established.

In order to prove the sufficiency part of the theorem, we use the following:

Lemma 3.2. *Assume that inequalities* (3.3) *hold, and let* $r_1 > 0$ *and* $\delta > 0$ *be chosen so that*

$$\xi \geq r_1 \Rightarrow g(\xi) \geq p_0 + 3\delta \tag{3.8}$$

and

$$\xi \leq -r_1 \Rightarrow g(\xi) \leq p_0 - 3\delta. \tag{3.9}$$

Let $L_1 > 0$ *be chosen so that if* $a \in (-\infty, \infty)$ *and* $T \geq L_1$, *then*

$$p_0 - \delta \leq \frac{1}{T} \int_a^{a+T} p(t)\, dt \leq p_0 + \delta. \tag{3.10}$$

If $y(t)$ *is a solution of* (2.3) *defined for* $t_1 \leq t \leq t_2$ *such that* $|y(t)| \geq r_1$ *and* $|y'(t)| \leq b$ *for* $t_1 \leq t \leq t_2$, *then*

$$t_2 - t_1 \leq \max\{L_1, (c/\delta)(|y(t_1)| - r_1), 2b/\delta\}.$$

Proof. We prove the lemma for the case $y(t) \geq r_1$ for $t_1 \leq t \leq t_2$; the proof of the case $y(t) \leq -r_1$ for $t_1 \leq t \leq t_2$ will then be clear.

Suppose then that $y(t) \geq r_1$ and $|y'(t)| \leq b$ for $t_1 \leq t \leq t_2$ and, contrary to the assertion of the lemma,

$$t_2 - t_1 > \max\{L_1, (c/\delta)(y(t_1) - r_1), 2b/\delta\}. \tag{3.11}$$

From (3.8) it follows that $g(y(t)) \geq p_0 + 3\delta$ for $t_1 \leq t \leq t_2$. Therefore, integrating the differential equation (2.3) from t_1 and t_2, and using (3.8), we obtain

$$\begin{aligned} c(y(t_2) - y(t_1)) &= y'(t_1) - y'(t_2) + \int_{t_1}^{t_2} p(t)\, dt - \int_{t_1}^{t_2} g(y(t))\, dt \\ &\leq 2b + (t_2 - t_1)(p_0 + \delta) - (t_2 - t_1)(p_0 + 3\delta). \end{aligned}$$

Since $2b < (t_2 - t_1)\delta$, it follows from (3.11) that

$$y(t_2) < y(t_1) - (t_2 - t_1)\delta/c < r_1,$$

which is a contradiction. This completes the proof of the lemma.

Lemma 3.3. *Assume that conditions* (3.3) *hold, and let* r_1, δ, *and* L_1 *be as in the statement of Lemma* 3.2. *If* $y(t)$ *is a solution of* (2.3) *such that* $y(t)$ *is defined for* $\bar{t} \leq t < \infty$, $|y(\bar{t})| \leq r_1$, *and* $|y'(t)| \leq b$ *for* $t \geq \bar{t}$, *then*

$$|y(t)| \leq r_1 + (b/2)\max\{L_1, 2b/\delta\}, \qquad t \geq \bar{t}. \tag{3.12}$$

Proof. If $[t_1, t_2]$ is a closed and bounded subinterval of $[\bar{t}, \infty)$ such that $|y(t_1)| = |y(t_2)| = r_1$, and $|y(t)| \geq r_1$ for $t_1 \leq t \leq t_2$, then according to Lemma 3.2 we have $t_2 - t_1 \leq \max\{L_1, 2b/\delta\}$. Let us consider the case where $y(t_1) = y(t_2) = r_1$, and $y(t) \geq r_1$ for $t_1 \leq t \leq t_2$. Let ξ be a number in the interval $[t_1, t_2]$ such that $y(t)$ assumes its maximum on the interval $[t_1, t_2]$ at ξ. Since $|y'(t)| \leq b$ for $t \in [t_1, t_2]$, it follows from the mean value theorem that

$|y(\xi)-r_1| = |y(\xi)-y(t_1)| \le b(\xi - t_1)$ and $|y(\xi)-r_1| = |y(t_2)-y(\xi)| \le b(t_2-\xi)$. Since either $\xi - t_1 \le (t_2-t_1)/2$ or $t_2 - \xi \le (t_2-t_1)/2$, it follows that

$$|y(\xi)| \le r_1 + b(t_2-t_1)/2 \le r_1 + (b/2)\max\{L_1, 2b/\delta\}.$$

A similar argument shows that if $y(t_1) = y(t_2) = -r_1$, $y(t) \le -r_1$ for $t_1 \le t \le t_2$, and $y(t)$ assumes its minimum at ξ, $\xi \in [t_1, t_2]$, then

$$|y(\xi)| \le r_1 + (b/2)\max\{L_1, 2b/\delta\}.$$

This proves the lemma.

The sufficiency part of Theorem 3.1 now follows from the last lemma. For, if $t_0 \in (-\infty, \infty)$ and $y(t)$ is a solution of the initial value problem (2.4), then, according to Theorem 2.1, there exists a number b, $b > 0$, depending on y_0 and v_0, such that $|y'(t)| \le b$ for $t \ge t_0$. If (3.3) holds and $|y_0| \le r_1$, then

$$|y(t)| \le r_1 + (b/2)\max\{L_1, 2b/\delta\}$$

for all t, $t \ge t_0$, and we have established the existence of a solution of (2.3) bounded on $[t_0, \infty)$.

It remains to establish (3.5) and (3.6), assuming (3.3). *We assume that these conditions hold throughout the remainder of this section.* Suppose that $y(t)$ is a solution of (2.3) with $|y(t_0)| \le A$ and $|y'(t_0)| \le B$, where $A > 0$, $B > 0$ are arbitrary. According to Theorem 2.1 there exists $b(A, B)$ such that $|y'(t)| \le b(A, B)$ for all $t \ge t_0$. We consider two cases.

Case I. $A \le r_1$. In this case, by Lemma 3.3,

$$|y(t)| \le r_1 + \frac{b(A, B)}{2}\max\left\{L_1, \frac{2b(A, B)}{\delta}\right\}, \qquad t \ge t_0, \tag{3.13}$$

and (3.5) holds. By Theorem 2.1, there exists a number k, independent of A and B, and a number $T(A, B)$ such that $|y'(t)| \le k$ for all $t \ge t_0 + T(a, b)$. Referring to the proof of Lemma 3.3, we see that if $[t_1, t_2]$ is a closed subinterval of $[t_0, \infty)$ such that $|y(t)| \ge r_1$ for $t \in [t_1, t_2]$, then

$$t_2 - t_1 \le \max\{L_1, 2b(A, B)/\delta\}.$$

Therefore, there exists a number $\bar{t}$ such that $|y(\bar{t})| \le r_1$ and

$$t_0 + T(A, B) \le \bar{t} \le t_0 + T(A, B) + \max\{L_1, 2b(A, B)/\delta\}.$$

Since $|y'(t)| \le k$ for $\bar{t} \le t < \infty$, it follows from Lemma 3.3 that

$$|y(t)| \le r_1 + (k/2)\max\{L_1, 2k/\delta\} = k_1, \qquad t \ge \bar{t}. \tag{3.14}$$

Therefore, if we set

$$T_0(A, B) = T(A, B) + \max\{L_1, 2b(A, B)/\delta\},$$

then (3.6) holds with k_1 defined as above.

Case II. $r_1 < A$. Assume that $r_1 < |y(t_0)| \le A$. According to Lemma 3.2, there exists a number t_* such that

$$(3.15)\qquad \begin{aligned} t_0 < t_* &\le t_0 + \max\{L_1, c(|y(t_0)| - r_1)/\delta, 2b(A, B)/\delta\} \\ &\le t_0 + \max\{L_1, c(A - r_1)/\delta, 2b(A, B)/\delta\}, \end{aligned}$$

and $|y(t_*)| \le r_1$.

A repetition of the argument given in Case I shows that if k_1 is defined by (3.14), then $|y(t)| \le k_1$ and $|y'(t)| \le k$ if

$$t \ge t_* + T(A, B) + \max\{L_1, 2b(A, B)/\delta\}.$$

Therefore, if we set

$$\begin{aligned} T_0(A, B) = T(A, B) &+ \max\{L_1, 2b(A, B)/\delta\} \\ &+ \max\{L_1, c(A - r_1)/\delta, 2b(A, B)/\delta\} \end{aligned}$$

then (3.6) holds. If $|y(t_0)| \le r_1 < A$, then, as shown in Case I, $|y(t)| \le k_1$ and $|y'(t)| \le k$ for

$$t \ge t_0 + T(A, B) + \max\{L_1, 2b(A, B)/\delta\},$$

and hence (3.6) will obviously hold with the preceding definition of $T_0(A, B)$.

If $r_1 < y(t_0) \le A$ and t is defined as above, then since $|y'(t)| \le b(A, B)$ for $t \ge t_0$, it follows from (3.15) that

$$\begin{aligned} |y(t)| &\le |y(t_0)| + (t_* - t_0)b(A, B) \\ &\le A + b(A, B)\max\{L_1, c(A - r_1)/\delta, 2b(A, B)/\delta\} \end{aligned}$$

for $t_0 \le t \le t_*$. It follows from Lemma 3.3 that (3.13) holds for $t \ge t^*$. Therefore, if we set

$$a(A, B) = A + b(A, B)\max\{L_1, c(A - r_1)/\delta, 2b(A, B)/\delta\},$$

then (3.5) holds for $t \ge t_0$.

If $|y(t_0)| \le r_1 < A$, then, since the right-hand side of inequality (3.13) is less than $a(A, B)$, (3.5) will still hold for $t \ge t_0$. This completes the proof of Theorem 3.1.

Remark. By examining the proofs in this section, one notes that if in Theorem 3.1 the assumption

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} p(t)\,dt = p_0$$

exists for all a uniformly in a is replaced by the assumption that

$$(3.16)\qquad \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\,dt$$

exists, then if (2.1) holds, (3.2) is necessary for the existence of a solution bounded on $[t_0, \infty)$; while if (2.1) is replaced by (3.4), and (3.16) holds, then

(3.2) is both necessary and sufficient that *all* solutions be bounded on $[t_0, \infty)$. However, (3.1) is needed in the next section.

4. Existence of a solution bounded on $(-\infty, \infty)$

Theorem 4.1. *Assume that* p *and* g *are continuous and that* (2.1), (2.2) *and* (3.1) *hold. If* (3.3) *holds, then there exists a solution* $y(t)$ *of* (2.3) *such that* $y(t)$ *and* $y'(t)$ *are both bounded on* $(-\infty, \infty)$.

Proof. For brevity we set $a_1 = a(1, 1)$, $b_1 = b(1, 1)$ in (3.5). For each integer n, $n = 1, 2, \ldots$, let $y_n(t)$ be the solution of the initial value problem

$$\begin{cases} y''(t) + cy'(t) + g(y(t)) = p(t), \\ y(-n) = 0, \qquad y'(-n) = 0. \end{cases} \tag{4.1}$$

According to (3.5), we have

$$|y_n(t)| \le a, \qquad |y_n'(t)| \le b \quad \text{for } t \ge -n. \tag{4.2}$$

For convenience we define a C^1-function $z_n(t)$ on $(-\infty, \infty)$ for $n = 1, 2, \ldots$ by setting

$$z_n(t) = \begin{cases} y_n(t), & -n \le t, \\ 0, & t < -n. \end{cases}$$

From (4.1), (4.2) and the fact that $|p(t)| \le c_1$ for $t \in (-\infty, \infty)$ we see that the sequences $\{z_n(t)\}_1^\infty$ and $\{z_n'(t)\}_1^\infty$ are uniformly bounded on $(-\infty, \infty)$, and for each n, $n = 1, 2, \ldots$, z_n has a piecewise continuous second derivative, which is bounded on $(-\infty, \infty)$ independently of n. It follows from Ascoli's lemma and induction that for each integer m, $m = 1, 2, \ldots$, there exists a sequence of functions $\{z_{m,n}(t)\}_{n=1}^\infty$ such that $\{z_{1,n}(t)\}_{n=1}^\infty$ is a subsequence of $\{z_n(t)\}_1^\infty$, $\{z_{m,n}(t)\}_{n=1}^\infty$ is a subsequence of $\{z_{m-1,n}(t)\}_{n=1}^\infty$ for $m > 1$, and for $m \ge 1$ both of the sequences $\{z_{m,n}(t)\}_{n=1}^\infty$ and $\{z_{m,n}'(t)\}_{n=1}^\infty$ converge uniformly with respect to t on the interval $-m \le t \le m$. If we denote the k th member of the diagonal sequence $\{z_{n,n}(t)\}_1^\infty$ by $w_k(t)$, $k = 1, 2, \ldots$, then for any t_1 and t_2, with $-\infty < t_1 \le t_2 < \infty$, both of the sequences $\{w_k(t)\}_1^\infty$ and $\{w_k'(t)\}_1^\infty$ converge uniformly with respect to t on $[t_1, t_2]$. We note that if $t_1 \in (-\infty, \infty)$ and $-k < t_1$, then w_k is of class C^2 on the interval $[t_1, \infty)$, and

$$w_k''(t) + cw_k'(t) + g(w_k(t)) = p(t)$$

on this interval. Thus, it follows that the sequence $\{w_k''(t)\}_1^\infty$ converges uniformly on bounded subintervals of $(-\infty, \infty)$. Therefore, if we define $w(t) = \lim_{k\to\infty} w_k(t)$ for $t \in (-\infty, \infty)$, then $w \in C^2(-\infty, \infty)$ and

$$w''(t) + cw'(t) + g(w(t)) = p(t).$$

Since, according to (4.2), we have $|w(t)| \le a$ and $|w'(t)| \le b$ for $t \in (-\infty, \infty)$, the proof of the theorem is complete.

5. The effect of large damping

In this section we only assume that g is defined and continuous on some interval. As a by-product of some of the estimates used in the previous sections we obtain

Theorem 5.1. *Let g be defined and continuous on some interval, and let $p(t)$ satisfy the conditions of Theorem 3.1. Assume that there exists an interior point ξ_0 of the domain of g such that $g(\xi_0) = p_0$, $g(\xi) > p_0$ for $\xi - \xi_0$ small and positive, and $g(\xi) < p_0$ for $\xi_0 - \xi$ small and positive. Given $\alpha > 0$, there exist numbers $\gamma_1 > 0$, $\gamma_2 > 0$ and $c^* > 0$ such that if $c \geq c^*$, $t_0 \in (-\infty, \infty)$, and $y(t)$ is a solution of*

$$y''(t) + cy'(t) + g(y(t)) = p(t) \tag{5.1}$$

satisfying $|y(t_0) - \xi_0| < \gamma_1$ and $|y'(t_0)| < \gamma_2$, then

$$|y(t) - \xi_0| < \alpha, \qquad |y'(t)| < \alpha \tag{5.2}$$

for $t \geq t_0$.

Proof. It suffices to prove the theorem under the additional assumptions that $p_0 = 0$ and $\xi_0 = 0$. For, we note that if $p_1(t) \equiv p(t) - p_0$ and $g_1(\xi) \equiv g(\xi + \xi_0) - p_0$, then

$$\lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} p_1(t)\,dt = 0 \tag{5.3}$$

uniformly with respect to $a \in (-\infty, \infty)$, $p_1(t)$ is continuous for $-\infty < t < \infty$, $g_1(0) = 0$, $g_1(\xi) > 0$ for ξ small and positive, and $g_1(\xi) < 0$ for ξ small and negative. Moreover, $y(t)$ is a solution of (5.1) satisfying (5.2) on $t_0 \leq t < \infty$ if and only if $z(t) \equiv y(t) - \xi_0$ is a solution of

$$z''(t) + cz'(t) + g_1(z(t)) = p_1(t)$$

satisfying $|z(t)| < \alpha$ and $|z'(t)| < \alpha$ for $t_0 \leq t < \infty$.

Suppose then that the conditions of Theorem 5.1 hold and, in addition, $\xi_0 = p_0 = 0$. Given $\alpha > 0$, choose $r_1 > 0$ so small that $2r_1 \leq \alpha$, $g(\xi) > g(0) = 0$ for $0 < \xi \leq 2r_1$, and $g(\xi) < 0$ for $-2r_1 \leq \xi < 0$. Let $g^*(\xi)$ be the function defined on $(-\infty, \infty)$ by

$$g^*(\xi) = \begin{cases} g(2r_1) & \text{if } \xi \geq 2r_1, \\ g(\xi) & \text{if } |\xi| < 2r_1, \\ g(-2r_1) & \text{if } \xi < -2r_1. \end{cases}$$

If $\delta > 0$ and $L_1 > 0$ are chosen so that $\xi \geq r_1 \Rightarrow g^*(\xi) \geq 3\delta$, $\xi \leq -r_1 \Rightarrow g^*(\xi) \leq -3\delta$, and

$$T \geq L_1 \Rightarrow -\delta \leq \frac{1}{T} \int_a^{a+T} p(t)\,dt \leq \delta$$

for all $a \in (-\infty, \infty)$, then, according to Lemma 3.3, for any solution $y(t)$ of the differential equation

$$y''(t) + cy(t) + g^*(y(t)) = p(t) \tag{5.4}$$

which satisfies the conditions $|y(t_0)| \le r_1$ and $|y'(t)| \le b$, we have

$$|y(t)| \le r_1 + (b/2)\max\{L_1, 2b/\delta\} \tag{5.5}$$

for all $t \ge t_0$.

Let $\gamma_2 > 0$ be chosen so small that $\gamma_2 < \alpha$ and

$$(\gamma_2/2)\max\{L_1, 2\gamma_2/\delta\} < r_1. \tag{5.6}$$

If c_1 and c_2 are constants such that $|p(t)| \le c_1$ for $-\infty < t < \infty$, $|g^*(\xi)| \le c_2$ for $-\infty < \xi < \infty$, and if we set $c_3 = c_1 + c_2$, then, referring to the proof of Theorem 2.1 (Case I), we see from (2.9) that if $y(t)$ is a solution of (5.4) defined on $[t_0, \infty)$, then

$$|y'(t)| \le \max\{|y'(t_0)|, c_3/c\}$$

for $t \ge t_0$. It follows that if c^* is so large that $c_3/c^* < \gamma_2$, if $c \ge c^*$, and $y(t)$ is any solution of (5.4) such that $|y'(t_0)| < \gamma_2$, then $|y'(t)| < \gamma_2$ on $[t_0, \infty)$.

We claim that if γ_2 and c^* are as above and we set $\gamma_1 = r_1$, then the assertion of Theorem 5.1 holds. To see this, we note that if $c \ge c^*$ and $\hat{y}(t)$ is a solution of (5.1) with $|\hat{y}(t_0)| < \gamma_1$ and $|\hat{y}'(t_0)| < \gamma_2$, then $\hat{y}(t)$ is also a solution of (5.4) for $t > t_0$ as long as $|\hat{y}(t)| \le 2r_1$. However, as shown above, any solution $y(t)$ of (5.4) with $|y(t_0)| < \gamma_1 = r_1$ and $|y'(t_0)| < \gamma_2$ satisfies $|y'(t)| \le \gamma_2$ for $t > t_0$ and, therefore, by (5.5) and (5.6),

$$|y(t)| \le r_1 + (\gamma_2/2)\max\{L_1, 2\gamma_2/\delta\} < 2r_1$$

for $t > t_0$. This shows that $\hat{y}(t)$ is defined for $t \ge t_0$, $|\hat{y}(t)| < 2r_1 \le \alpha$, and $|\hat{y}'(t)| \le \gamma_2 < \alpha$ for $t \ge t_0$. This proves the theorem.

Using the same type of argument which was used to prove Theorem 4.1, one can prove

Theorem 5.2. *If the hypotheses of Theorem* 5.1 *hold, then given any* $\alpha > 0$ *there exists a number* $c^* > 0$ *such that for* $c \ge c^*$ *there exists a solution* $y(t)$ *of* (5.1) *defined on* $(-\infty, \infty)$ *such that* $|y(t) - \xi_0| < \alpha$ *and* $|y'(t)| < \alpha$ *for all* t *in* $(-\infty, \infty)$.

6. An example

We conclude with an example to illustrate the difference in behavior which can occur when the damping constant c increases from small positive values to large positive values. Let g be an odd C^1 function such that $g'(\xi) \ge 0$ for all $\xi \in (-\infty, \infty)$, $g(\xi) = b\xi$ for $-d \le \xi \le d$, where $b > 0$ and $d > 0$, and such that the $\lim_{\xi\to\pm\infty} g(\xi) = g(\pm\infty)$ are finite. In engineering literature, g

is referred to as a saturation function since the corresponding restoring force acts as a linear spring for displacements within a certain range, but *saturates*, i.e. becomes nearly constant, for large displacements.

An elementary analysis shows that all solutions of

$$y''(t) + g(y(t)) = 0 \tag{6.1}$$

are periodic and that the period is a nondecreasing function of the amplitude (see [7]). This implies that there exists a nonconstant periodic solution $y_0(t)$ of (6.1) of period L, $L > 0$, such that all L-periodic solutions of the variational equation

$$z''(t) + g'(y_0(t))z(t) = 0$$

are constant multiples of $y_0'(t)$ (see [7]). The theory developed in [7] shows that if $f(t)$ is a continuous L-periodic function such that

$$\int_0^L y_0'(t)f(t)\,dt = 0 \tag{6.2}$$

and

$$\int_0^L y_0''(t)f(t)\,dt \neq 0 \tag{6.3}$$

then for ε sufficiently small and positive and c sufficiently small and positive (*the magnitude depending* on ε) there exists an L-periodic solution of

$$y''(t) + cy'(t) + g(y(t)) = \varepsilon f(t) \tag{6.4}$$

which is close to $y_0(t)$ in the C^1-sense. A particular f that works is $f(t) = y_0''(t)$. The same result guarantees, for suitably restricted positive ε and c, the existence of an L-periodic solution of (6.4) close to $-y_0(t)$ in the C^1-sense. Finally, for c and ε small and positive, there exists a small amplitude solution of (6.4) which is actually a solution of the linear differential equation

$$y''(t) + cy'(t) + by(t) = \varepsilon f(t).$$

Thus, for suitable positive constants c and ε, (6.4) has at least two L-periodic solutions.

In contrast, we now show that if $p(t)$ satisfies the hypotheses of Theorem 3.1—as any continuous periodic or almost periodic function will—and g is the saturation function described above, then if $|p_0| < bd$ and c is sufficiently large and positive, there exists a unique solution of (2.3) bounded on $(-\infty, \infty)$, which is asymptotically stable and globally attracting.

Suppose then that $|p_0| < bd$ and let ξ_0 be the unique point in $(-d, d)$ such that $g(\xi_0) = p_0$. Let $p_1(t) = p(t) - p_0$ and $g_1(\xi) = g(\xi + p\xi_0) - p_0$. Then (5.3) holds, $g_1(0) = 0$, and g_1 is linear on an interval centered at 0. Moreover, there is a one-to-one correspondence between solutions of

$$z''(t) + cz'(t) + g_1(z(t)) = p_1(t) \tag{6.5}$$

by means of the transformation $y(t) = z(t) - \xi_0$.

Let $r_1 > 0$ be so small that $g_1(\xi)$ is linear in the range $-2r_1 \le \xi \le 2r_1$, and let $\delta > 0$ be so small that $g_1(\xi) > 3\delta$ if $\xi \ge r_1$ and $g_1(\xi) < -3\delta$ if $\xi \le -r_1$. Let $L_1 > 0$ be so large that

$$-\delta < \frac{1}{T}\int_a^{a+T} p_1(t)\,dt < \delta$$

if $a \in (-\infty, \infty)$ and $T \ge L_1$.

Let c_2 be the supremum of $|g_1(\xi)|$ on $(-\infty, \infty)$, and let $c_3 = c_1 + c_2$. According to (2.10), if $z(t)$ is a solution of (6.5) such that $|z(t_0)| \le B$, then there exists $T(B)$ such that if $t \ge t_0 + T(B)$ then $|z(t)| \le 2c_3/c \equiv k$.

Let $c^* > 0$ be so large that $(c_3/c^*)\max\{L_1, 4c_3/c^*\} < r_1$, and suppose that $c > c^*$. It follows from (3.14) that, given $A > 0$ and $B > 0$, if $z(t)$ is a solution of (6.5) such that $|z(t_0)| \le A$ and $|z'(t_0)| \le B$ for some t_0 in $(-\infty, \infty)$, then there exists $T_0(A, B)$ such that for $t \ge t_0 + T_0(A, B)$ we have $|z'(t)| \le k$ and

$$|z(t)| \le r_1 + (k/2)\max\{L_1, 2k/\delta\} < 2r_1.$$

Thus it follows that if $z(t)$ is a solution of (6.5) which is bounded on $(-\infty, \infty)$ and $c > c^*$, then $|z(t)| < 2r_1$ for all t in $(-\infty, \infty)$. Therefore, since $g_1(\xi) = b\xi$ for $|\xi| < 2r_1$, $z(t)$ is a solution of the linear differential equation

$$z''(t) + cz'(t) + bz(t) = p_1(t). \tag{6.6}$$

Since the homogeneous differential equation

$$z''(t) + cz'(t) + bz'(t) = 0 \tag{6.7}$$

has no solution bounded on $(-\infty, \infty)$ other than $z \equiv 0$, it follows from standard theory (see, e.g., [4, Chapter IV]) that (6.6) has a unique solution bounded on $(-\infty, \infty)$. Therefore, for $c \ge c^*$ it follows that (6.5), and hence (2.3), has a unique solution bounded on $(-\infty, \infty)$.

Let $c \ge c^*$, and let $z_0(t)$ be the unique solution of (6.5) bounded on $(-\infty, \infty)$. If $A > 0$, $B > 0$, and $\hat{z}(t)$ is another solution of (6.5) such that $|\hat{z}(t_0)| \le A$ and $|\hat{z}'(t_0)| \le B$ for some $t_0 \in (-\infty, \infty)$, then, as shown above, $|\hat{z}(t)| \le 2r_1$ for $t \ge t_0 + T_0(A, B)$; thus $z_0(t)$ and $\hat{z}(t)$ are solutions of (6.6) for $t \ge t_0 + T_0(A, B)$. Since the difference of these two solutions is a solution of (6.7), and since the zero solution of (6.7) is globally asymptotically stable, we see that $\hat{z}(t) - z_0(t) \to 0$ as $t \to \infty$. This establishes our claim that the unique solution of (2.3), which is bounded on $(-\infty, \infty)$, is globally asymptotically stable.

References

1. J. Bebernes and M. Martelli, *Periodic solutions for Liénard systems*, Boll. Un. Mat. Ital. A (5) **16** (1979), 398–405.
2. A. S. Besicovitch, *Almost periodic functions*, Cambridge Univ. Press, 1935.

3. W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath, 1965.
4. P. O. Frederickson and A. C. Lazer, *Necessary and sufficient damping in a second order oscillator*, J. Differential Equations **5** (1969), 262–270.
5. J. K. Hale, *Ordinary differential equations*, Wiley, 1969.
6. A. C. Lazer, *On Schauder's fixed point theorem and forced second-order nonlinear oscillations*, Math. Anal. Appl. **21** (1968), 421–425.
7. W. S. Loud, *Periodic solutions of* $x'' + cx' + g(x) = \varepsilon f(t)$, Mem. Amer. Math. Soc. No. 31 (1959).
8. N. Rouche and J. Mawhin, *Equations différentielles ordinaires. Stabilité et solutions periodiques*, Masson, Paris, 1973 (translated by Pitman, 1980).
9. G. E. H. Reuter, *Boundedness theorems for nonlinear differential equations of the second order*, J. London Math. Soc. **27** (1952), 48–58.
10. G. Sansone and R. Conti, *Nonlinear differential equations*, Macmillan, 1964.
11. J. R. Ward, Jr., *Periodic solutions of a class of ordinary differential equations*, Proc. Amer. Math. Soc. **78** (1980), 350–352.
12. T. Yoshizawa, *Stability theory of Liapunov's second direct method*, Math. Soc. Japan, 1966.

DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCE, AND STATISTICS, UNIVERSITY OF TEXAS AT SAN ANTONIO, SAN ANTONIO, TEXAS 78285-0664